Proof that if an integer is not even, it is odd:

Assumptions made: the difference of integers is an integer.

To begin, I will define an even number n as one where there is some integer k such that n=2k. An integer n is even if and only if there is some k such that n=2k. An odd number is one where there is some integer k such that n=2k+1. An integer n is odd if and only if there is some integer k such that n=2k+1.

(Sub)proof that every integer is either even or odd:

We can start by observing that 0 is even because there is a k such that 2*k=0 (in this case, k=0).

In the case that some n is even, n=2k. This means that n+1 is odd because n+1=2k+1. In the case that n is odd, n=2k+1. This means that n+1=2k+2=2(k+1). Therefore, adding 1 to an odd integer results in an even integer and adding 1 to an even integer results in an odd integer. This chain logically follows forever and winds up including every integer.

Subproof that 1 is not even. Suppose 1 is even. This would mean that there is some integer k such that 2k=1. But there is no such integer and so 1 is not even.

Proof that a number cannot be both even and odd:

Suppose that integers can be odd and even and that there is some integer n that is both even and odd. This means that there is some integer n, which is both even and odd, which means that there are integers k and j such that n=2k=2j+1. This means that 2k=2j+1, which means that 2k-2j=1. Doing some rearranging results in 2(k-j)=1. K-j=1/2, but since the difference of integers is always an integer, k-j cannot equal ½ if both k and j are integers. This means that no such integers k and j exist such that n=2k=2j+1.

An alternative way of proving this last part. 2(k-j)=1. If (k-j) equaled an integer, then 1 would have to be even, but since 1 is not even, according to our proof from earlier, then k-j is not equal to an integer.

This proves that a number cannot be both even and odd. But since I've already shown that every integer is either even or odd, then an integer that is not even must be odd.