Another Proof of the Odd-Even LemmaBy Paul Sanders

In order to understand this proof, it is required that the reader can actually read, count, and also be able to follow the rules of a simple game. My aim is to convince anyone who has had at least a first grade education, and enough of an attention span to read three pages, that *odd* and *even* numbers are mutually exclusive. In other words, it is impossible to have an integer that is both odd and even

But enough about boring numbers, integers, and large words. Let's get on to the game! To play the game, we only need a collection of assorted items (coins, dice, gold doubloons, etc). For our game, I have a bag of small stones we will use.

The Stone-Pulling Game:

- The stone-pulling game is a 2-person game (you and me)
- The gameplay is as follows:
  - The players start the game by sitting down across from each other. ne player decides who goes first in the game
  - Second, the other player places a number of stones from the bag in the middle between the two players. (this player cannot choose zero stones)
  - Next, he player that was selected to go first begins by taking a stone from the middle and placing it on their right, thus creating their own pile. This action is called a turn.
  - Next, the other player has their turn, taking a stone from the middle and starting a pile at their respective right.
  - Repeat the previous two steps, this is called a round of the game (or round)
  - Continue to play rounds until no stones remain in the middle. At this point, the game ends and the winner is declared.

- The player whose turn allows them to place the last remaining stone from the middle into their respective pile wins the game
- Players cannot pass on a turn, meaning you must take a stone from the middle every time it is your opportunity to do so. Players cannot put stones back in the middle after they are removed.
- Players must take one and only one stone from the middle on their turn.

Sounds easy enough, right? Let's play together, you and me! Since you're still new at this, I'll let you have the easy job of being the player who decides which of us goes first. I'll handle the more difficult job of taking the stones out of the bag, OK? Oh you wanna go first? Awesome, sounds good!

(*We play several games, always with you going first. You lose every game)*

Huh? You want *me* to go first now? Hahaha I get it, you think you keep losing because *you're* going first. Nope! That's not what is going on, but lets keep playing and see if you get lucky!

(*We play several more games, but this time I go first every time. You still lose every single game)*

What? Yeah we can stop for a little while. Oh, you're starting to get frustrated, right? You're wondering "How can I always lose? No matter who goes first I always lose!"

Well, I have not been totally honest with you….I'm rigging the game. I'm not exactly cheating per-se (because that would be most unpleasant for a UVA student), but rather exploiting a problem in the design of the game that allows me to always win. I'll show you how it works.

Basically, the fact that you decide who goes first *then* I put the stones in is the key to the whole fix. Let's break the game down to understand what is going on: You say who goes first, then I put some stones in, and we pull stones until there are none left in the middle. But even further, let's break down what is happening every round (Remember, a round is when *you* take a stone, and *I* take a stone). Every round, one stone leaves the middle and goes to your pile, and one stone goes to mine. For simplicity's sake, I'm going to define this occurrence as a *pair*.

<u>Definition</u>: A **pair** is when two stones leave the middle on a given round, with one stone going to one player's pile, and the other stone going to the other players pile.

<u>Definition:</u> When the number of stones in the middle allow for a pair to be made every round of the game, we call that number **totally paired** (or completely paired).

<u>Definition</u>: If it is impossible to make a pair in the last round of a game, we call that round **unfinished** or **incomplete**

Whoof! that's more definitions than I wanted, but I think this will do the trick. The definitions might seem way too formal at this point, and I agree with you, it is a bit much. But they are necessary to see what I am doing. Case in point, did you consider that there are some numbers of stones that *cannot* be totally paired? It's true! In the case of games where the player who goes first wins, the stones are never able to be totally paired. Think about it! Consider a game with only one stone in the middle. Since there is only one stone in the middle, there cannot be a pair that round by definition. Thus that game results in an incomplete round and the player who goes first for that round will always win.

Also, any game that ends in an incomplete round will also ensure victory to the first player. In this case, being impossible to make a pair on the last round means that the player who goes first in the round will pick up the last stone, and thus win. I should also convince you it is possible to have a middle pile start with a number of more than one stones, and still result in an incomplete round at the end. Think about a game with three stones(you can count, so you should have an idea of three). The game would have two rounds: one where a pair is made, and one incomplete round. You can keep extending this line of thought with pairs and rounds up to five stones, seven stones, any arbitrary number of stones really. To continue this pattern of victory to the person who goes first, you just have to be careful to ensure that the number of stones you chose can never be totally paired.

On the other hand, f the number of stones in the middle can be totally paired, then the player who goes first will always lose. Imagine a game where there is only one round, and one pair is made (so there was more that one stone in the middle, see above case). In this game, the player who goes second will always wi because when it is their turn to take a stone, they are also finishing the pair. If there was no stone for the second player to take, then the round would be considered incomplete. So if a pair can be made every round, the second player will win! This is the definition of a totally paired number of stones. This shows that totally paired number of stones can never result in a game with an incomplete round!

The fact that *I* put the number of stones in *after* knowing who will go first, gives me a huge advantage. If you decide to go first, I make sure the number of stones in the middle *can* be totally paired, and I win. If you force me to go first I make sure the number of stones *cannot* be totally paired, thus resulting in an incomplete round and I win.

How do I make the game fair, you ask? Well, it depends on what you mean by fair, but one way to do it would be to change the way information is revealed in the game. Instead of you picking who goes first then me placing the stones down, we should both write down our respective choices on paper at the same time and then show each other. That gets rid of the extra information I was able to use to win every game. (The proof of this is left as an exercise to the reader.)

Bringing this back around to our original odd-even discussion: a number can be called *even* if it would result in a totally paired game of stone-pulling. A number can be called *odd* if it would result in an incomplete round at the end of a game of stone-pulling. Since we have demonstrated that it is impossible for a totally paired number of stones to result in an incomplete round at the end, *odd* and *even* are mutually exclusive. A game that starts with an *odd* number of stones in the middle will ensure victory to the person who goes first, while a game that starts with an *even* number of stones will ensure victory to the person who goes second. Thus *odd* and *even* can be defined AND shown to be mutually exclusive without the need of algebraic expressions! Huzzah!

There are ways in which I could extend this stone-pulling demonstration to highlight the difference between odd and even numbers further, even extending it to the case of negative integers. However, I promised I was only going to make you read a three page proof, so I'll stop here for now. Thank you for your time!