Challenge problem 6 (PS3Prob11) solution by induction

The following is a proof by induction that the longest satisfiable formula of v variables contains $({}_{v}C_{3})$*7 clauses:

First we take a base case of v = 3. The maximum number of clauses in the longest satisfiable formula for three variables is 7, which includes every possible 3-variable clause, minus one. This is equal to $({}_{3}C_{3})$*7, so the proposition holds for v = 3.

To determine the number of clauses in the longest satisfiable formula of v+1 variables, we can add the total number of additional possible clauses of v variables, minus the minimum number of clauses necessary to remove to make the equation satisfiable for the combination of inputs for which the original equation was satisfiable. **We know that the result is the longest satisfiable formula because for any unique clause we add back into the formula, we know that it will evaluate false for the satisfiable formula's only satisfying assignment.**

It may be unclear whether there is a longer solution which may be obtained by adding more clauses than are removed to our final solution here.
To add clauses that we've excluded due to satisfiability, clauses would have to be removed so that the formula is satisfiable for a different combination of inputs (because it would no longer be satisfied for the original satisfying input combination, as stated above). This would apply asymmetry, though (because that would mean that by selecting a particular input combination, we can get a longer formula), which doesn't exist because the calculation we performed for the number of input combinations applies to *any* input combination for which the original formula was satisfied.

To calculate the total number of new clause possibilities for v+1, we need every possible two-variable combination which may be selected from v variables. The reason for this is that every new clause, to be considered new, *must* have the (v+1)'th variable in it, or it would be a possibility available to the original formula of v variables. There are therefore only 2 available "slots" per clause into which we must add one of the v remaining variables. The number of available combinations is thus $({}_{v}C_{2})$.

The number of possible inversions of these variables (including the new variable) is 2 to the power of the number of variables per clause, in our case 2^3. By multiplying this by the number of clause combinations, we get the total number of newly-available clauses to choose from when forming the longest satisfiable formula of v+1 variables: $2^3 *({}_{v}C_{2})$.

What's left is to calculate the minimum number of clauses to subtract from this number to make the new formula continue to be satisfiable. Because the satisfying assignment is fixed (it has to be the same assignment from the original formula of v variables, because that is the only combination for which it was satisfied), the inversions of the variables must be fixed. The clause combinations are calculated the same as above however, and are equal to $({}_{v}C_{2})$.
The final formula for the number of additional clauses is:

$2^3 *({}_{v}C_{2}) - ({}_{v}C_{2}) = ({}_{v}C_{2})(2^3 -1) = 7*({}_{v}C_{2})$.

We add this formula to the formula for the length of a v-variable satisfiable formula.

$7*({}_{v}C_{2}) + 7*({}_{v}C_{3}) = 7*({}_{v}C_{2} + {}_{v}C_{3}) = 7*((v)*(v-1)/2 + (v)*(v-1)*(v-2)/6) =$
$7*(((3)*(v)*(v-1) + (v)*(v-1)*(v-2))/6) = 7*((v)*(v-1)*(v-2+3)/6) = 7*((v)*(v-1)*(v+1)/6) = 7*({}_{v+1}C_{3})$

Therefore, if the proposition holds for a formula of v variables, it also holds for a formula of v+1 variables, and the proposition is therefore true for all numbers of variables greater than or equal to three.