Class 17: Infinite Sets

Schedule

Before Thursday, everyone should have finished reading MCS Chapter 8.

Problem Set 7 is due **Friday (27 Oct) at 6:29pm**.

Infinite Sets

Finite Cardinality. The *cardinality* of the set

$$\mathbb{N}_k = \{n | n \in \mathbb{N} \wedge n < k\}$$

is k. If there is a *bijection* between two sets, they have the same cardinality. (Class 9)

Does this definition tell us the cardinality of $\mathbb{N}$?

Definition. A set S is *infinite* if there is no bijection between S and any set $\mathbb{N}_k$ (as defined above).

Show that $\mathbb{Z}$ is infinite.

Cardinality of Infinite Sets

Equal Carinalities. We say $|A| = |B|$ for arbitrary sets A, B (and say that they have the same cardinality), if there is a bijection between A and B.

Comparing Cardinalities. We say $|B| \leq |A|$ for arbitrary sets A, B (and say that B's cardinality is less than or equal to the cardinality of A), if there is a *surjective function* from A to B.

Show that $|A| = |B|$ implies $|B| \leq |A|$ and $|A| \leq |B|$. Be careful as these sets might not be finite, in which case we cannot simply use natural numbers to denote their cardinalities.

Schröder-Bernstein Theorem: If $|A| \leq |B|$ and $|B| \leq |A|$, then there is a bijection between A and B, namely $|A| = |B|$. (Not proven in cs2102; this is somewhat tricky to prove! For a full proof, see the linked lecture notes.)

Other Definitions for Infinite Sets

Dedekind-Infinite. A set A is *Dedekind-infinite* if and only if there exists a *strict subset* of A with the same cardinality as A. That is,

$$\exists B \subset A. \exists R. \, R \text{ is a bijection between } A \text{ and } B.$$

Recall the definition of strict subset:

$$B \subset A \iff B \subseteq A \wedge \exists x \in A \, . \, x \notin B.$$

Third Definition. A set S is *third-definition infinite* if $|S| \geq |\mathbb{N}|$ (as defined on the previous page). Namely, there is a *surjective function* from S to $\mathbb{N}$.

Are the above three definitions of (standard) *infinite* and *Dedekind-infinite* and *third-definition infinite* equivalent definitions?

Definition. A set S is *countable* if and only if there exists a surjective function from $\mathbb{N}$ to S. (That is, ≤ 1 arrow out from $\mathbb{N}$, ≥ 1 arrow in to S.) Using our notation defined above, this means $|S| \leq |\mathbb{N}|$.

Prove that these sets are countable: $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}$ (rationals), $\emptyset$, $\mathbb{N} \cup (\mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N} \times \mathbb{N})$, all finite sequences of elements of $\mathbb{N}$.

Definition. A set S is *countably infinite* if and only if it is *countable* and it is *infinite* (according to standard definition).

Must a *countable* set that is *Dedekind-infinite* be *countably infinite*?

Using the definition of countable, and third definition of infinite, show that S is countably infinite if and only if there is a bijection between S and $\mathbb{N}$. (We might as well use this definition in the future.)

