Exam 1

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Name: ______________________________
Email ID: ______________________________

For this exam, you must **work alone**. You are not permitted to obtain help from people other than asking clarifying questions of the course staff. You are not permitted to provide help to others taking the exam. **You may not use any resources other than the one page of notes you prepared.**

Sign below to indicate that you understand these expectations and can be trusted to behave honorably:

Signed: ____

The exam has 8 questions. For each question, there is ample space provided to hold an excellent answer. If you need more space, you can use the backs of pages, but include clear markings and arrows to indicate the answer that should be graded. We will assume anything not inside and answer box or clearly marked from one, is your scratch work that should not be considered in scoring your answers.

We use the following notations throughout the exam:

	Logical Operators
$P \implies Q$	Logical implication: when P is **T**, Q must be **T**
$P \iff Q$	Double implication (if and only if): $(P \implies Q) \land (Q \implies P)$
$\text{NOT}(P)$, $\neg P$, $\overline{P}$	Logical negation
$P \land Q$	Logical conjunction (**T** when both P and Q are **T**)
$P \lor Q$	Logical disjunction (**T** when P or Q is **T** (or both)
	Quantifiers
$\forall x \in A.\, P(x)$	$P(x)$ for *every* element x of the set A
$\exists x \in A.\, P(x)$	$P(x)$ for *at least one* element x of the set A
	Set Operators
$x \in A$	Set membership, A contains the element x
$x \notin A$	Set non-membership, A does not contain x
$A \subseteq B$	A is a subset of B: $\forall x \in A . x \in B$.
$A = B$	Set equality: $(A \subseteq B) \land (B \subseteq A)$
$A \cup B$	Set union: $\forall x . x \in A \cup B \iff (x \in A) \lor (x \in B)$.
$A \cap B$	Set intersection: $\forall x . x \in A \cap B \iff (x \in A) \land (x \in B)$.
$A - B$	Set difference: $\forall x . x \in A - B \iff x \in A \land x \notin B$.
$\overline{A}$	Set complement: $\forall x . x \in D . x \in \overline{A} \iff x \notin A$.
$A \times B$	Cartesian product: $\forall a \in A, b \in B.\, (a, b) \in A \times B$.

Do not open past this page until instructed to do so.

Inference Rules

1. For each candidate inference rule below, indicate if it is *sound* or *unsound* (circle the correct answer). For the rules that are unsound, provide a counterexample to show it is unsound. You do not need to provide any justification for the rules that are sound.

   a. $\dfrac{P}{Q}$

   Circle one: *Sound* *Unsound*
   Justification (if unsound):

   b. $\dfrac{P \wedge (P \implies Q)}{Q}$

   Circle one: *Sound* *Unsound*
   Justification (if unsound):

   c. $\dfrac{P \wedge Q}{P \vee Q}$

   Circle one: *Sound* *Unsound*
   Justification (if unsound):

   d. $\dfrac{\overline{Q} \implies \overline{P}}{P \implies Q}$

   Circle one: *Sound* *Unsound*
   Justification (if unsound):

   e. $\dfrac{P \wedge \overline{P}}{P \implies \overline{P}}$

   Circle one: *Sound* *Unsound*
   Justification (if unsound):

Satisfiability

2. For each formula below, determine if it is *satisfiable* and if it is *valid*. The definitions (from Class 4) are restated below:

   Definition: valid. A logical formula is *valid* if there is no way to make it **false**. That is, no matter what truth values its variables have, it is always **true**.

   Definition: satisfiable. A logical formula is *satisfiable* if there is *some* way to make it **true**. That is, there is at least one assignment of truth value to its variables that makes the forumla true.

   Justifications are *optional*. A correct answer with no justification will be worth full expected credit; an incorrect answer with no justification is worth nothing, but *an incorrect answer with a justification that reveals some understanding may be worth some credit.*

   a. $(P \vee \overline{P})$

   Circle one: *Satisfiable* or *Not Satisfiable*

   Circle one: *Valid* or *Not Valid*

   Justification (optional):

   b. $(P \vee Q) \wedge (\overline{P} \vee Q)$

   Circle one: *Satisfiable* or *Not Satisfiable*

   Circle one: *Valid* or *Not Valid*

   Justification (optional):

   c. $(P \implies Q) \vee (Q \implies P)$

   Circle one: *Satisfiable* or *Not Satisfiable*

   Circle one: *Valid* or *Not Valid*

   Justification (optional):

Logical Formula

3. Show convincingly that P IMPLIES Q is logically equivalent to $\overline{P} \vee Q$.

Well Ordered Sets

4. Explain why the set of the integers ($\mathbb{Z}$) is not well ordered by $<$. (Expected answers will give a good intuitive reason; better-than-expected answers will provide a convincing proof.)

Relations

5. R is a total ([$\geq$ 1 arrow out]), injective ([$\leq$ 1 arrow in]), relation between A and B with graph $G \subseteq (A \times B)$. For each statement below, indicate if it *must be true*, *might be true* (could be either true or false), or *cannot be true* (must be false). Provide a short justification supporting your answer.

   a. $|A| \leq |B|$

   Circle one: *Must be True* or *Might Be True* or *Cannot Be True*

   Justification:

   b. $A = \emptyset \implies B = \emptyset$

   Circle one: *Must be True* or *Might Be True* or *Cannot Be True*

   Justification:

   c. $B = \emptyset \implies A = \emptyset$

   Circle one: *Must be True* or *Might Be True* or *Cannot Be True*

   Justification:

   d. R^{-1} (the inverse relation of R) is a surjective function.

   Circle one: *Must be True* or *Might Be True* or *Cannot Be True*

   Justification:

Proofs

6. Define the sets $People$ and UVA as:

   $People$ ::= all people in the universe
   UVA ::= set of all students at UVA

   Assume these two axioms:

   1. $\forall s \in UVA.\, Honorable(s)$
   2. $\forall p \in People.\, Honorable(p) \implies \neg(Cheats(p) \lor Lies(p) \lor Steals(p))$

   Prove that if $p \in People$ and $Cheats(p)$, then p must not be a UVA student.

7. Below is a bogus proof that claims to prove every integer greater than 6 can be written as $3a + 5b$ for natural numbers a and b ($a, b \in \mathbb{N}$). Identify the first incorrect inference step, and explain clearly why it is wrong.

   a. We state the proposition as,

   $$P(n) ::= \exists a, b \in \mathbb{N}.\, n = 3a + 5b$$

   and prove $\forall n \in \mathbb{N}, n \geq 6.\, P(n)$.

   b. We prove using the well-ordering principle.

   c. Define the set of counter-examples, C:

   $$C ::= \{n \in \mathbb{N}, n \geq 6 | \forall a, b \in \mathbb{N}.\, n \neq 3a + 5b\}$$

   d. Assume C is non-empty.

   e. By well-ordering principle, there must be some minimum element of C, $m \in C$.

   f. We reach a contradiction by showing $P(m)$.

   g. Since m is the minimum element of C, we know $\forall k \in \mathbb{N}, 6 \leq k < m.\, P(k)$.

   h. We know $m > 6$ since $n \geq 6$ and $P(6)$ is true: $6 = 3 \cdot 2 + 5 \cdot 0$.

   i. Since $m - 3 < m$, this implies $P(m - 3)$.

   j. $P(m - 3)$ implies $\exists a, b \in \mathbb{N}.\, m - 3 = 3a + 5b$.

   k. So, $m = 3a + 5b + 3 = 3(a + 1) + 5b = 3a' + 5b$ for some $a' \in \mathbb{N}$.

   l. This shows $P(m)$, which is a contradiction since we selected $m \in C$. Hence, C must be empty, proving that $P(n)$ holds for all $n \in \mathbb{N}, n \geq 6$.

Incorrect step:

What is wrong?

8. Prove by induction that every finite non-empty subset of the integers contains a *greatest* element, where an element $x \in S$ is defined as the greatest element if $\forall z \in S - \{x\}.\, x > z$.

End of Exam Questions (last page is optional feedback).

Optional Feedback

This question is optional and will not negatively effect your grade.

Do you feel your performance on this exam will fairly reflect your understanding of the course material so far? If not, explain why. (Feel free to provide any other comments you want on the exam, the course so far, or your hopes for the rest of the course here.)

Problem	1	2	3	4	5	6	7	8	Total
Score									

