Final Exam

Read this page and fill in your name, pledge, and email ID now.

Name: ______________________________

Email ID: ______________________________

For this exam, you must **work alone**. You are not permitted to obtain help from people other than asking clarifying questions of the course staff. You are not permitted to provide help to others taking the exam.

You may not use any resources other than the one page of notes you prepared. You should turn in those notes with your exam. Make sure your name is clearly written on the notes page in case it gets separated from your exam.

Sign below to indicate that you understand these expectations and can be trusted to behave honorably:

Signed: ______________________________

The exam has 13 questions. For each question, there is ample space provided to hold an excellent answer. If you need more space, you can use the backs of pages, but include clear markings and arrows to indicate the answer that should be graded. We will assume anything not inside the answer space or clearly marked from it, is your scratch work that should not be considered in scoring your answers.

Do not open past this page until instructed to do so.

Logical Formulas and Inference Rules

1. For each candidate inference rule below, indicate if it is *sound* or *unsound* (circle the correct answer). For the rules that are unsound, provide a counter-example to show it is unsound. You do not need to provide any justification for the rules that are sound.

   a. $\dfrac{P}{Q}$

   Circle one: *Sound* *Unsound*
   Counter-example (if unsound):

   b. $\dfrac{(P \land Q) \lor (P \land \overline{Q})}{P}$

   Circle one: *Sound* *Unsound*
   Counter-example (if unsound):

   c. $\dfrac{(P \land Q) \land (P \land \overline{Q})}{P}$

   Circle one: *Sound* *Unsound*
   Counter-example (if unsound):

   d. $\dfrac{(P \implies Q) \land Q)}{P}$

   Circle one: *Sound* *Unsound*
   Counter-example (if unsound):

Well Ordering

2. For each set and operator below, answer if the set is *well-ordered* or not. Support your answer with a brief, but clear and convincing, argument.

a. the even natural numbers; $<$.

Circle one: *Well-Ordered* *Not Well-Ordered*
Justification:

b. the non-negative real numbers; $<$.

Circle one: *Well-Ordered* *Not Well-Ordered*
Justification:

c. the empty set; $<$.

Circle one: *Well-Ordered* *Not Well-Ordered*
Justification:

d. pow($\mathbb{N}$); compare$(S, T) := |S| < |T|$

Circle one: *Well-Ordered* *Not Well-Ordered*
Justification:

Sets and Relations

3. Indicate for each statement if it is valid (always true) or invalid. For invalid statements, provide a counter-example supporting your answer.

a. For any sets A and B, $|A \cup B| \leq |A| + |B|$.

Circle one: *Valid* *Invalid*
Counter-example (if invalid):

b. For any sets A and B, $(\forall a \in A.\, a \in B) \implies A \subseteq B$.

Circle one: *Valid* *Invalid*
Counter-example (if invalid):

c. For any sets A and B, there exists a total ([≥ 1 arrow out]), injective ([≤ 1 arrow in]) relation R between A and $A - B$.

Circle one: *Valid* *Invalid*
Counter-example (if invalid):

d. For any sets A and B, $B \in \text{pow}(A) \implies |B| < |A|$.

Circle one: *Valid* *Invalid*
Counter-example (if invalid):

Induction

4. Prove by induction that every finite non-empty subset of the real numbers contains a *least* element, where an element $x \in S$ is defined as the least element if $\forall z \in S - \{x\}.\, x < z$. (Note: you should not just assume all finite sets are well ordered for this question.)

State Machines

5. Consider the state machine, $M_1 = (S, G, q_0)$ below:

$$
\begin{aligned}
S &= \{(a, b) \mid a, b \in \mathbb{N}\} \\
G &= \{(a, b) \rightarrow (a', b') \mid a, a', b, b' \in \mathbb{N} \wedge a + b = a' + b'\} \\
q_0 &= (0, 0)
\end{aligned}
$$

a. Describe the reachable states for M_1.

For each of the following predicates, answer whether or not it is a *preserved invariant* for M_1 as defined above, and provide a brief justification supporting your answer.

b. $P(q = (a, b)) := a > b$

Circle one: *Preserved Invariant* or *Not Preserved*

Justification:

c. $P(q = (a, b)) := a + b$ is odd

Circle one: *Preserved Invariant* or *Not Preserved*

Justification:

Cardinality

6. For each set defined below, answer is the set if *countable* or *uncountable* and support your answer with a convincing and concise proof. (Recall that $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers.)

a. set of all subsets of students in cs2102

b. $\{(a, b) | a, b \in \mathbb{N}\}$

c. $\{(a, b) | a \in \mathbb{N}, b \in \mathbb{R}\}$

d. the set of all Turing Machines that accept no strings

Proofs

Consider the Take-Away game: start with n sticks; at each turn, a player must remove 1 or 2 sticks. The player who takes the last stick wins.

7. Prove that Player 1 has a winning strategy for a two-player game of Take-Away where Player 1 moves first if the initial number of sticks is n is not divisible by 3.

Program Correctness

Consider the Python program below, that returns `True` if and only if none of the elements of the input list are below 5. You may assume `p` is a non-empty list of natural numbers.

```
def all_good(p):
   i = 0
   good = True
   while i < len(p):
      if p[i] < 5:
         good = False
      i = i + 1
   return good
```

8. Complete the definition of the state machine, $M_g = (S, G, q_0)$, below that models `all_good`.

$$
\begin{aligned}
S &= \{(i, g) \,|\, i \in \mathbb{N}, g \in \{\textbf{True}, \textbf{False}\}\} \\
G &= \{(i, g) \rightarrow (i', g') \mid \\
&\qquad i, i' \in \mathbb{N}, g, g' \in \{\textbf{True}, \textbf{False}\} \\
&\qquad \wedge \underline{\qquad} < \underline{\qquad\qquad\qquad\qquad\qquad} \\
&\qquad \wedge i' = \underline{\qquad\qquad\qquad\qquad\qquad} \\
&\qquad \wedge g' = \begin{cases} \underline{\qquad} & \text{if p[i]} < 5 \\ \underline{\qquad} & \text{otherwise} \end{cases} \\
&\qquad \} \\
q_0 &= \underline{\qquad\qquad\qquad\qquad\qquad}
\end{aligned}
$$

9. Prove that for any input that is a finite list of natural numbers, the state machine M_g always terminates, and the final state is a state where the value of g is **True** if and only if the input list contains no elements with value below 5.

Recursive Data Types

Define a *BalancedTree* as:

– **Base case: null** $\in$ *BalancedTree*.

– **Constructor case:** if $t_1, t_2 \in$ *BalancedTree* and $\text{count}(t_1) = \text{count}(t_2)$ then $\text{node}(t_1, t_2) \in$ *BalancedTree*.

where count is defined for all *BalancedTree* objects as:

$$\text{count}(t) := \begin{cases} 0 & t = \textbf{null} \\ 1 + \text{count}(t_1) + \text{count}(t_2) & t = \text{node}(t_1, t_2) \end{cases}$$

The left and right operations are defined by:

$$\begin{aligned} \text{left}(\text{node}(t_1, t_2)) &:= t_1 \\ \text{right}(\text{node}(t_1, t_2)) &:= t_2 \end{aligned}$$

10. Explain why left is not a *total function* for *BalancedTree* objects domain.

11. Prove that for all non-null *BalancedTree* objects, t, $\text{count}(\text{left}(t)) = \text{count}(\text{right}(t))$.

Turing Machines

12. Prove that there exists a Turing Machine that (1) accepts an infinite number of inputs and (2) does not terminate on an infinite number of inputs.

Countable Sets are Well-Ordered

13. Prove that all countable sets are well-ordered.

End of Exam Questions (last page is optional).

Optional Guidance

This question is optional, and it will not count against you if you decline to answer it.

What grade do you believe you deserve in cs2102? ________________________

Explain why:

Anything else you want me to know?

Score: ______

